Digital Image

(B4M33DZO, Winter 2024)

Lecture 2:

Fourier Transform

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

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Fourier Transform

time$\{t\}$ $\Longleftrightarrow$ **frequency**$\{u\}$

Fourier Transform

$$\textbf{time}\{t\} \iff \textbf{frequency}\{u\}$$

forward: $$F(u) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i u t} \mathrm{d}t$$

inverse: $$f(t) = \int_{-\infty}^{\infty} F(u) \cdot e^{+2\pi i u t} \mathrm{d}u$$

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basis functions: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$

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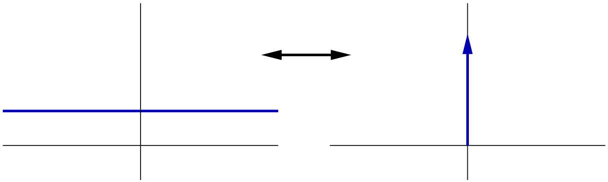
amplitude: $$|F(u)| = \sqrt{\mathbf{Re}(F(u))^2 + \mathbf{Im}(F(u))^2}$$

phase: $$\Phi(F(u)) = \tan^{-1}\left(\frac{\mathbf{Im}(F(u))}{\mathbf{Re}(F(u))}\right)$$

Fourier Transform

$$1 \quad \Longleftrightarrow \quad \delta(u)$$



Fourier Transform

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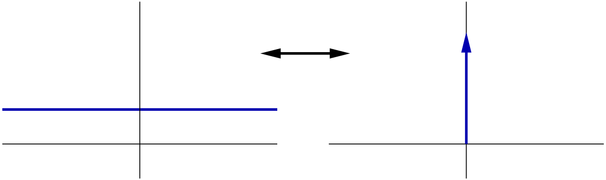
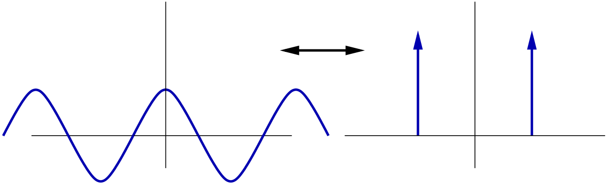
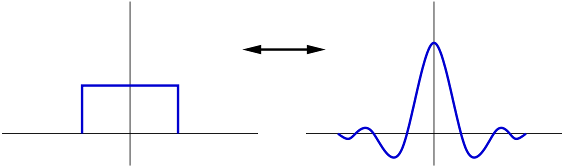


$$\cos(2\pi k x) \quad \Longleftrightarrow \quad \frac{1}{2}\left(\delta(u+k)+\delta(u-k)\right)$$



Fourier Transform

$$\mathbf{1}(x+k) - \mathbf{1}(x-k) \quad \Longleftrightarrow \quad \frac{1}{\pi u}\sin(2\pi k u)$$



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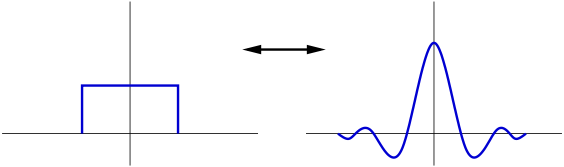
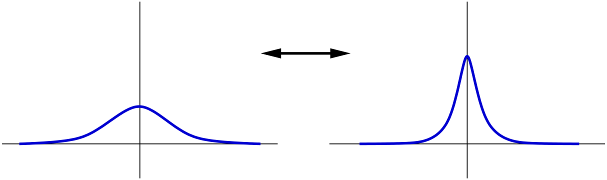


$$\exp\left(-kx^2\right) \quad \Longleftrightarrow \quad \sqrt{\frac{\pi}{k}}\exp\left(-\frac{\pi^2}{k}u^2\right)$$



Fourier Transform

Linearity:

$$a \cdot f(x) + b \cdot f(x) \iff a \cdot F(u) + b \cdot F(u)$$

Fourier Transform

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Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \mathrm{d}u$$

Fourier Transform

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Convolution theorem:

$$\int_{-\infty}^{\infty} f(x) \cdot g(t-x) \mathrm{d}x \quad \Longleftrightarrow \quad F(u) \cdot G(u)$$

Fourier Transform

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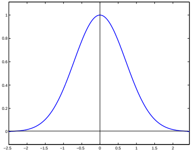
Shift theorem:

$$\int_{-\infty}^{\infty} f(x-a) \cdot e^{-2\pi jux} \mathrm{d}x = F(u) \cdot e^{-2\pi jua}$$

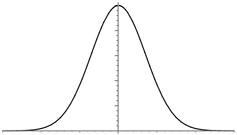
Fourier Transform

time | **derivatives** | **frequency**



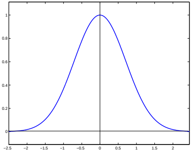
0th:

$$f(x) \Longleftrightarrow F(u)$$



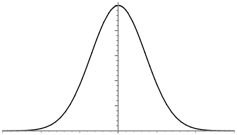
Fourier Transform

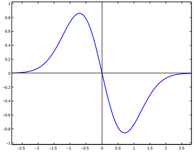
time | derivatives | frequency



0th:

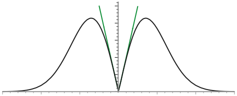
$$\color{red}{f}(x) \Longleftrightarrow \color{blue}{F}(u)$$



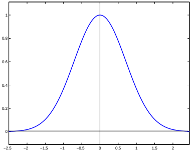
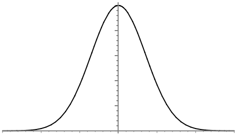
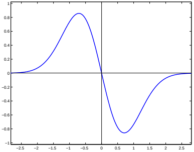
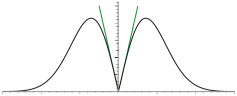
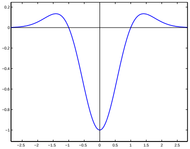
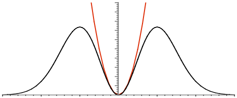


1st:

$$\color{red}{f'}(x) \Longleftrightarrow u \cdot \color{blue}{F}(u)$$



Fourier Transform

time	derivatives	frequency
	**0th:** $f(x) \Longleftrightarrow F(u)$	
	**1st:** $f'(x) \Longleftrightarrow u \cdot F(u)$	
	**2nd:** $f''(x) \Longleftrightarrow u^2 \cdot F(u)$	

Fourier Transform (in 2D)

position$\{x, y\}$ $\iff$ **frequency & orientation**$\{u, v\}$

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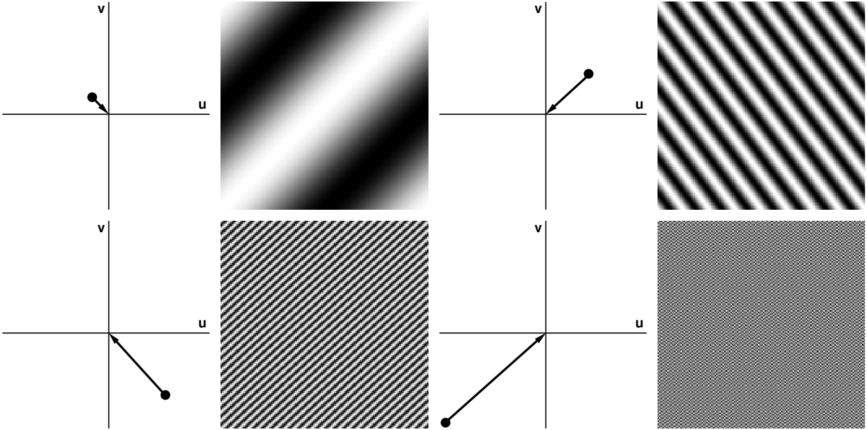
forward: $$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-2\pi i(ux+vy)}\mathrm{d}x\mathrm{d}y$$

inverse: $$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\cdot e^{+2\pi i(ux+vy)}\mathrm{d}u\mathrm{d}v$$

Fourier Transform (in 2D)

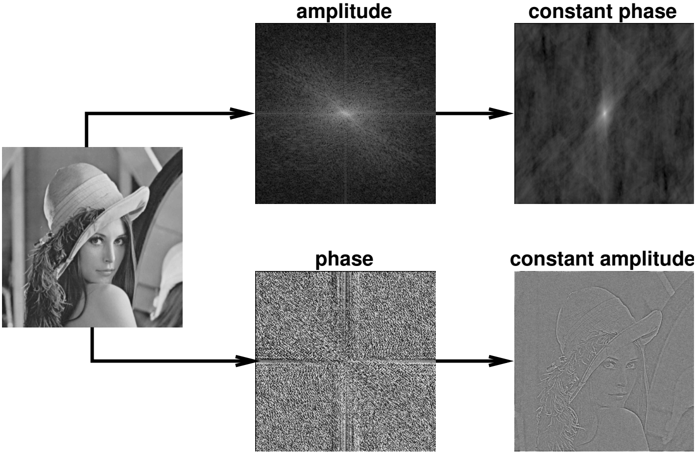
basis functions:



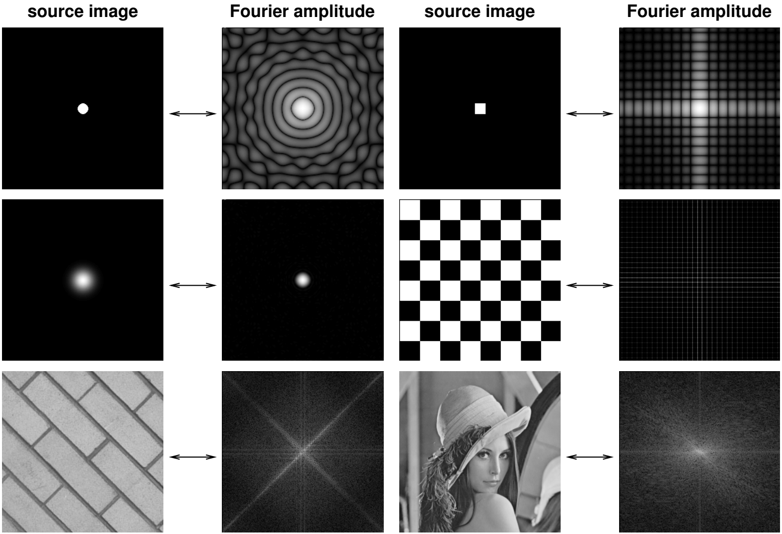
Fourier Transform (in 2D)

Edges with orientation $\arctan(v/u)$ **and frequency** $\sqrt{u^2+v^2}$**:**

Amplitude $\Rightarrow$ **intensity** **Phase** $\Rightarrow$ **"location"**



Fourier Transform (in 2D)



Fourier Transform (in 2D)

Discrete Fourier Transform (DFT):

forward: $$F[u,v] = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f[x,y] \cdot e^{-2\pi i (ux+vy)/N}$$

inverse: $$f[x,y] = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F[u,v] \cdot e^{+2\pi i (ux+vy)/N}$$

Computation complexity: $\mathcal{O}(N^4)$. **Can we do it faster?**

Fourier Transform (in 2D)

Fast Fourier Transform (FFT):

separability: $$F[u,v] = \frac{1}{N}\sum_{x=0}^{N-1} e^{-2\pi i ux/N} \cdot \left(\sum_{y=0}^{N-1} f[x,y] \cdot e^{-2\pi i vy/N} \right)$$

recursion: $$\begin{aligned} F[u] &= F_{\text{even}}[u] + F_{\text{odd}}[u] \cdot e^{-2\pi i u/N} \\ F[u+N/2] &= F_{\text{even}}[u] - F_{\text{odd}}[u] \cdot e^{-2\pi i u/N} \end{aligned}$$

Computation complexity: $\mathcal{O}(N^2 \log N)$

Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$

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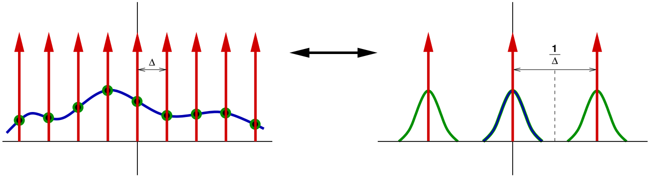
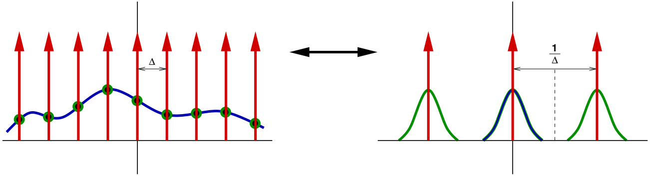


Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$



$$s(x) = \sum_k \delta\left(x - k\Delta\right) \iff S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

Sampling Theorem

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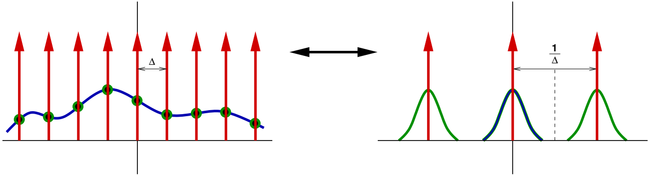
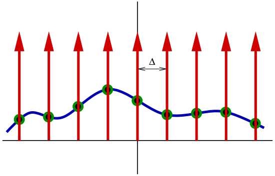


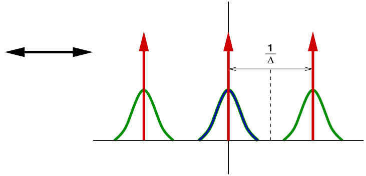
$$s(x) = \sum_k \delta(x - k\Delta) \iff S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

$$d(x) = f(x) \cdot s(x) \iff D(u) = (F * S)(u)$$

Sampling Theorem

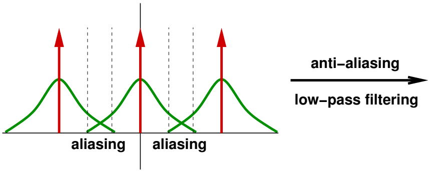
Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$

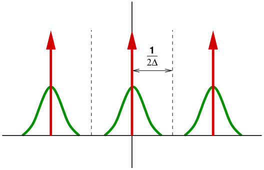




$$\color{red}{s}(x) = \sum_k \delta\left(x - k\Delta\right) \iff \color{red}{S}(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

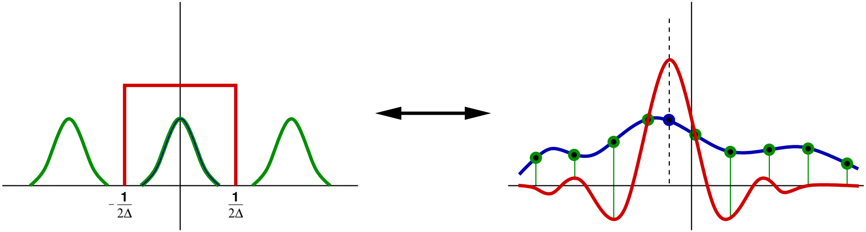
$$\color{green}{d}(x) = \color{blue}{f}(x) \cdot \color{red}{s}(x) \iff \color{green}{D}(u) = (\color{blue}{F} * \color{red}{S})(u)$$





Sampling Theorem

Signal reconstruction:



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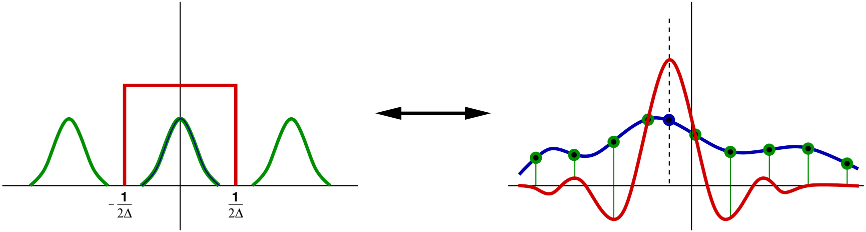

$$\color{red}{B}(u) = \mathbf{1}\left(u + \frac{1}{2\Delta}\right) - \mathbf{1}\left(u - \frac{1}{2\Delta}\right) \iff \color{red}{b}(x) = \frac{1}{\pi x}\sin\left(\frac{\pi x}{\Delta}\right)$$

Sampling Theorem

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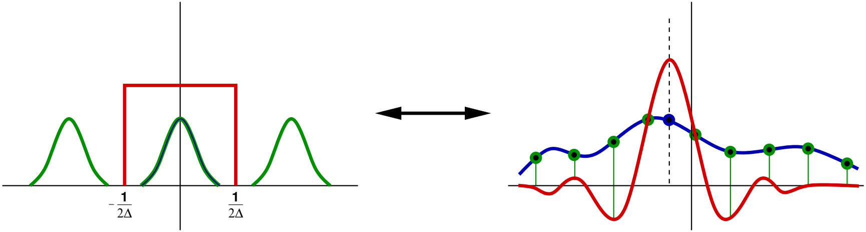

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$$F(u) = D(u) \cdot B(u) \iff f(x) = (d * b)(x)$$

Sampling Theorem

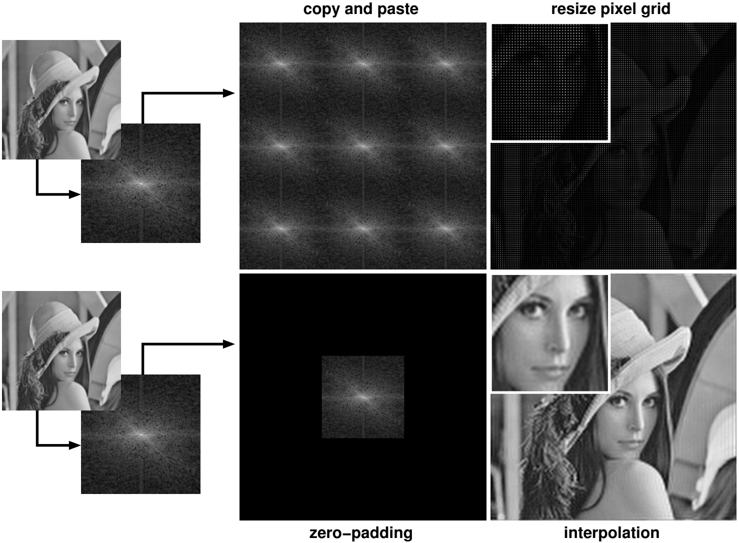

Image Resampling

Problem: convolution with sinc ⇒ time consuming, ringing artifacts.

Sinc approximations with narrow support (bicubic, bilinear, box):