Digital Image

(B4M33DZO, Winter 2024)

Lecture 7:

Image Deformation

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

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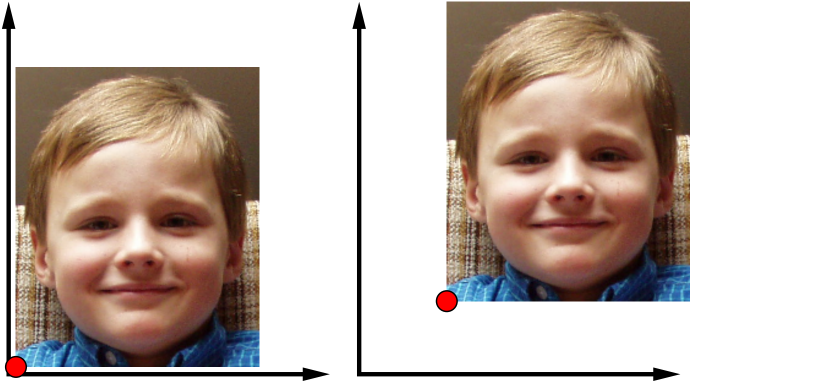
Translation

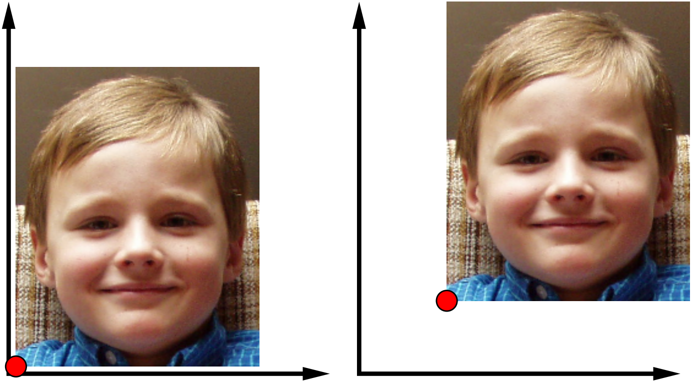
1-point:

$$x' = x + x_0$$
$$y' = y + y_0$$



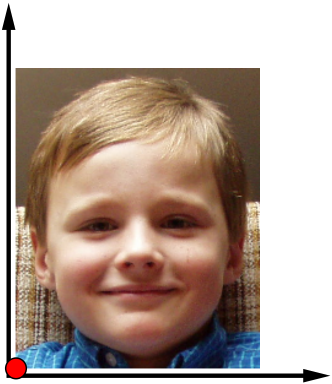
Translation

1-point: $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



Translation

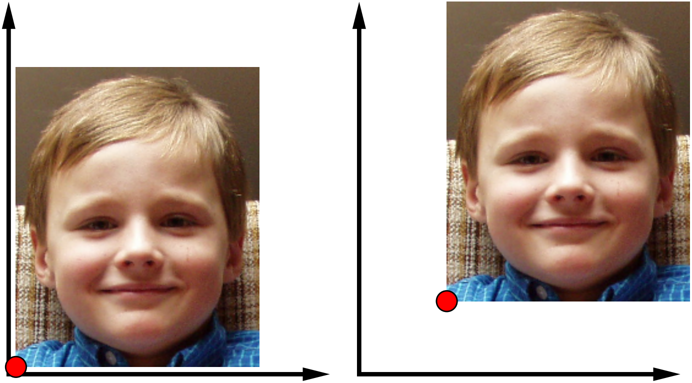
1-point: $\mathbf{x}' = \mathbf{x} + \mathbf{t}$



Translation

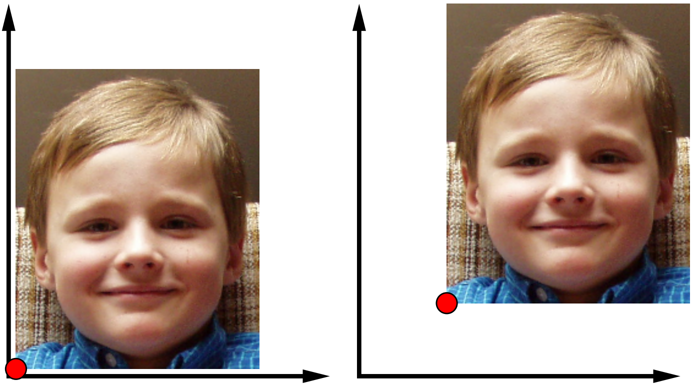
1-point:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



Translation

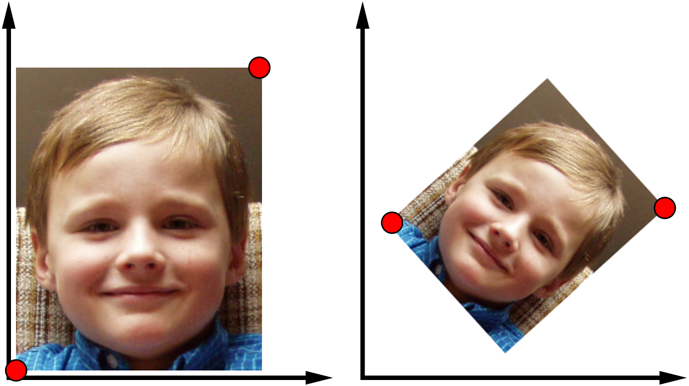
1-point: $\mathbf{x}' = \mathbf{T} \cdot \mathbf{x}$



Similarity

2-point: $x' = x + x_0$

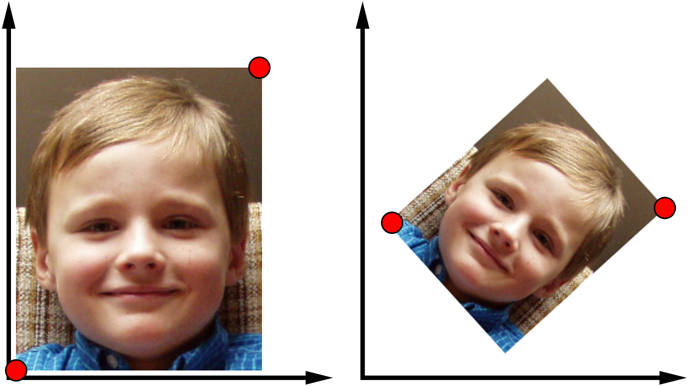
(translation) $y' = y + y_0$



Similarity

2-point: $x' = \mu x + x_0$

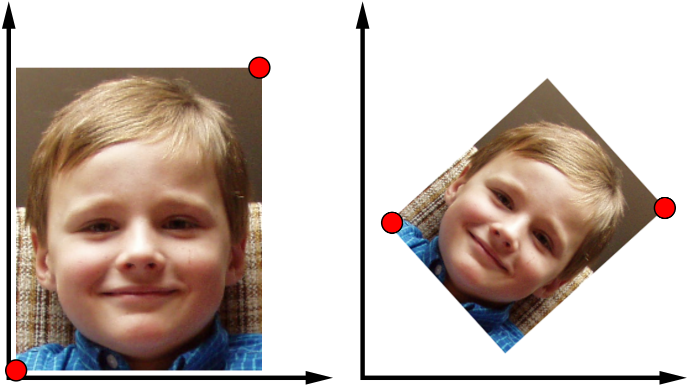
(translation + scale) $y' = \mu y + y_0$



Similarity

2-point:
(translation + scale + rotation)

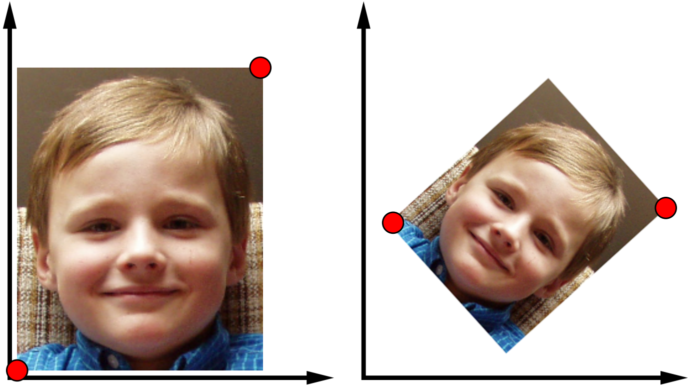
$$x' = \mu\cos(\alpha)x + \mu\sin(\alpha)y + x_0$$
$$y' = -\mu\sin(\alpha)x + \mu\cos(\alpha)y + y_0$$



Similarity

2-point:
(translation + scale + rotation)

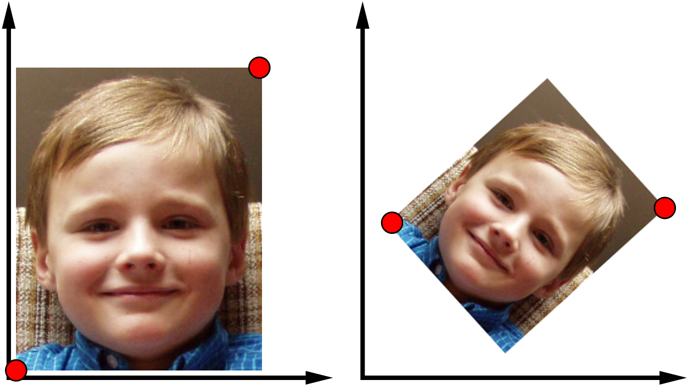
$$\mathbf{x}' = \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{R} \cdot \mathbf{x}$$



Similarity

2-point:
(translation)

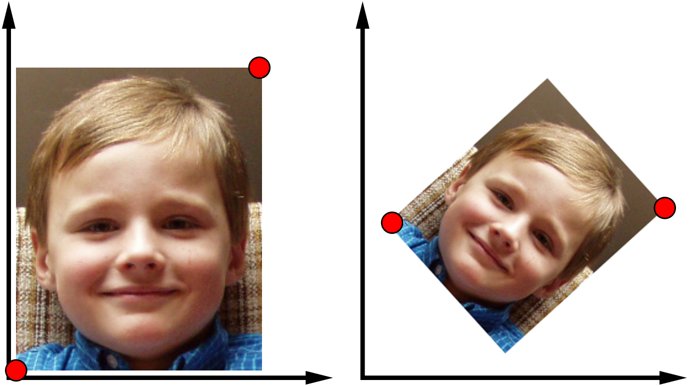
$$\mathbf{T} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



Similarity

2-point:
(scale)

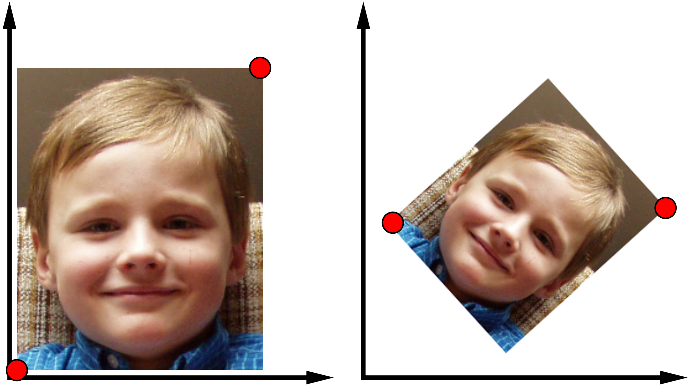
$$\mathrm{M} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Similarity

2-point:
(rotation)

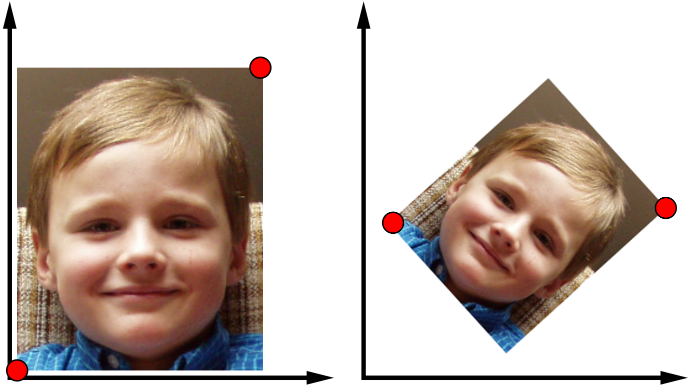
$$\mathbf{R} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Similarity

2-point:
(translation + scale + rotation)

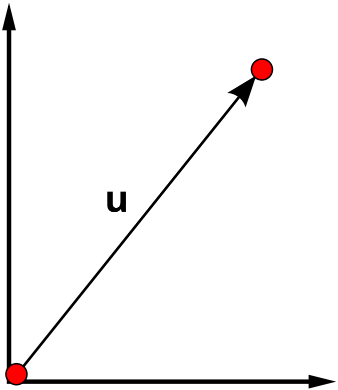
$$\mathbf{S} = \begin{pmatrix} \mu\cos(\alpha) & \mu\sin(\alpha) & x_0 \\ -\mu\sin(\alpha) & \mu\cos(\alpha) & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



Similarity

2-point:
(scale + rotation)

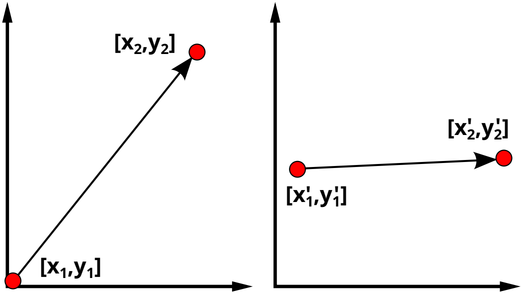
$$\mu = \frac{|\mathbf{v}|}{|\mathbf{u}|} \qquad \cos(\alpha) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}$$



Similarity

2-point:
(translation + scale + rotation)

$$x' = ax + by + x_0$$
$$y' = -bx + ay + y_0$$



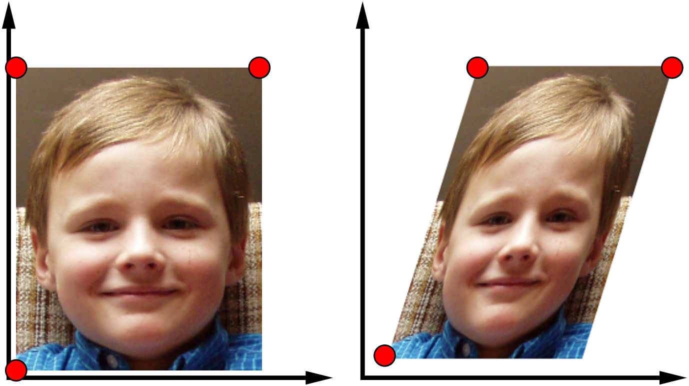
solve a linear system:

$$x'_1 = ax_1 + by_1 + x_0$$
$$y'_1 = -bx_1 + ay_1 + y_0$$
$$x'_2 = ax_2 + by_2 + x_0$$
$$y'_2 = -bx_2 + ay_2 + y_0$$

Affine

3-point:
(translation + scale + rotation + skew)

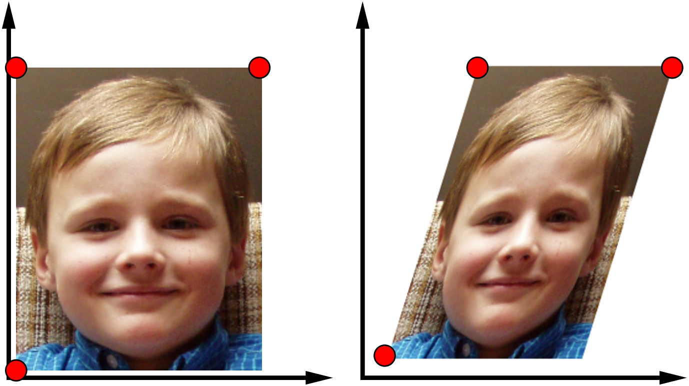
$$x' = a_{11}x + a_{12}y + x_0$$
$$y' = a_{21}x + a_{22}y + y_0$$



Affine

3-point:
(translation + scale + rotation + skew)

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

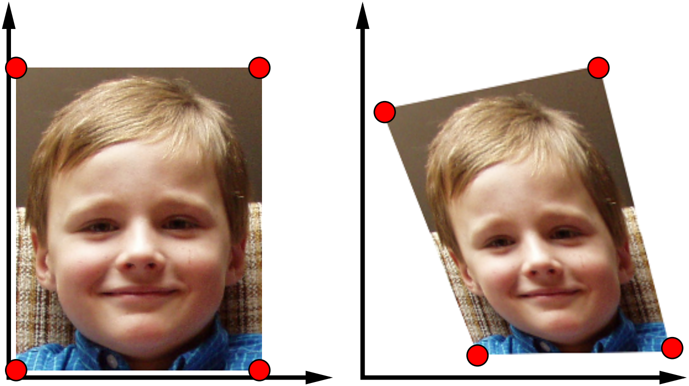


Projective

4-point:
(homography)

$$x' = \frac{a_{11}x + a_{12}y + x_0}{f_1 x + f_2 y + 1}$$

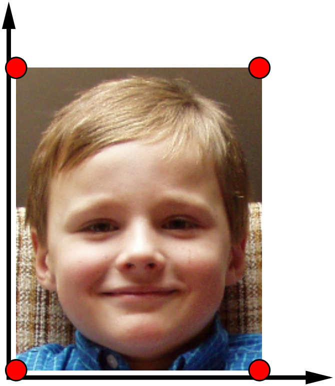
$$y' = \frac{a_{21}x + a_{22}y + y_0}{f_1 x + f_2 y + 1}$$

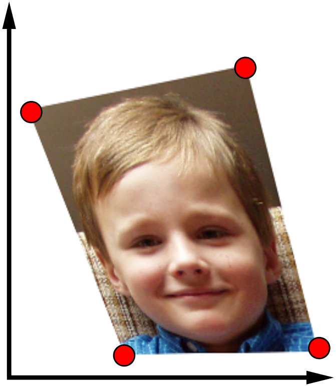


Projective

4-point: $(f_1 x + f_2 y + 1) \cdot x' = a_{11} x + a_{12} y + x_0$

(homography) $(f_1 x + f_2 y + 1) \cdot y' = a_{21} x + a_{22} y + y_0$

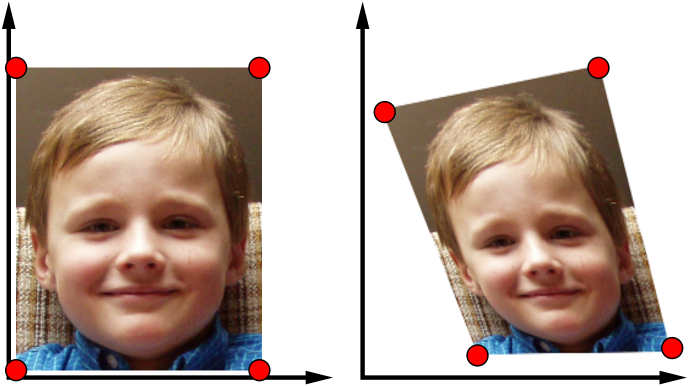




Projective

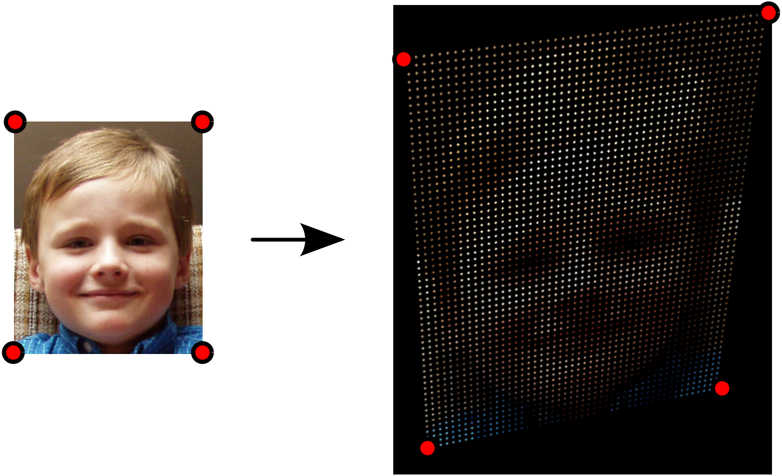
4-point:
(homography)

$$\mathbf{H} = \begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ f_1 & f_2 & 1 \end{pmatrix}$$

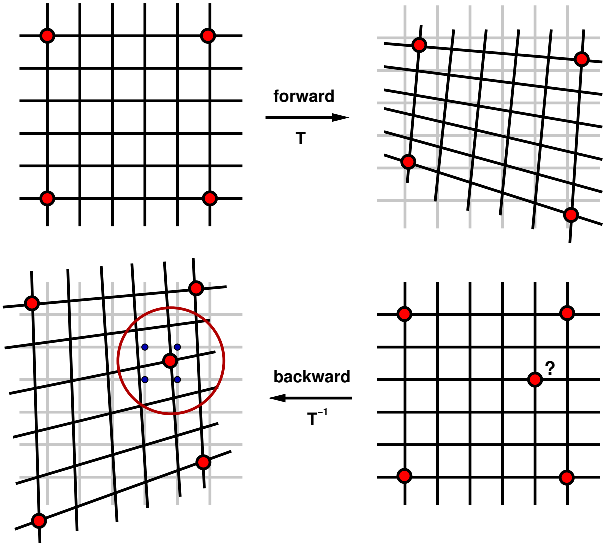


Forward mapping

Problem: directly transformed pixels will not fill the target grid.



Backward mapping



Signal reconstruction

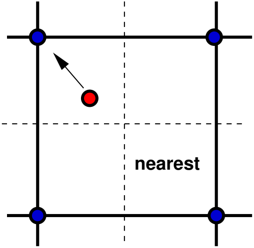
Nearest neighbor:

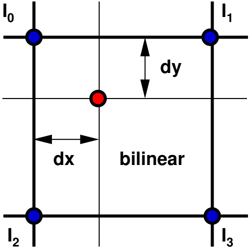
$$\mathrm{I}[x', y'] = \mathrm{I}[\lfloor x \rfloor, \lfloor y \rfloor]$$

Bilinear interpolation:

$$\begin{aligned} \mathrm{I}_0 &= \mathrm{I}[\lfloor x \rfloor, \lfloor y \rfloor] & \mathrm{I}_3 &= \mathrm{I}[\lfloor x \rfloor + 1, \lfloor y \rfloor + 1] \\ \mathrm{I}_1 &= \mathrm{I}[\lfloor x \rfloor + 1, \lfloor y \rfloor] & \mathbf{d}_x &= x - \lfloor x \rfloor \\ \mathrm{I}_2 &= \mathrm{I}[\lfloor x \rfloor, \lfloor y \rfloor + 1] & \mathbf{d}_y &= y - \lfloor y \rfloor \end{aligned}$$

$$\mathrm{I}[x', y'] = (\mathrm{I}_0(1 - \mathbf{d}_x) + \mathrm{I}_1 \mathbf{d}_x)(1 - \mathbf{d}_y) + (\mathrm{I}_2(1 - \mathbf{d}_x) + \mathrm{I}_3 \mathbf{d}_x)\mathbf{d}_y$$





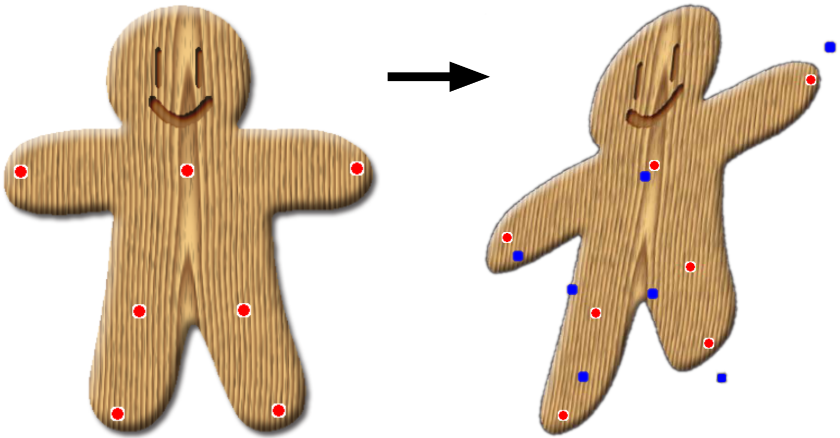
Free-form deformation

N-point:

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Free-form deformation

Least squares (affine):



Free-form deformation

Least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} |\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^{\mathbf{2}}$$

Free-form deformation

Least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} |\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^{\mathbf{2}}$$

$$\frac{\partial}{\partial \mathbf{t}} \sum_i |\mathbf{p}_i\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^2 = 0 \quad \Rightarrow \quad \mathbf{t}^* = \mathbf{q_c} - \mathbf{p_c}\mathbf{A}$$

Free-form deformation

Least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} |\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^{\mathbf{2}}$$

$$\frac{\partial}{\partial \mathbf{t}} \sum_i |\mathbf{p}_i\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^2 = 0 \quad \Rightarrow \quad \mathbf{t}^* = \mathbf{q_c} - \mathbf{p_c}\mathbf{A}$$

$$\mathbf{p}_c = \tfrac{1}{N}\textstyle\sum_i \mathbf{p}_i \quad \mathbf{q}_c = \tfrac{1}{N}\textstyle\sum_i \mathbf{q}_i$$

Least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} |\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^{\mathbf{2}}$$

$$\frac{\partial}{\partial \mathbf{t}} \sum_i |\mathbf{p}_i\mathbf{A} + \mathbf{t} - \mathbf{q_i}|^2 = 0 \quad \Rightarrow \quad \mathbf{t}^* = \mathbf{q_c} - \mathbf{p_c}\mathbf{A}$$

$$\mathbf{p}_c = \tfrac{1}{N}\sum_i \mathbf{p}_i \quad \mathbf{q}_c = \tfrac{1}{N}\sum_i \mathbf{q}_i$$

$$\hat{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{p}_c \quad \hat{\mathbf{q}}_i = \mathbf{q}_i - \mathbf{q}_c$$

Free-form deformation

Least squares (affine):

$$\mathbf{A}^* = \arg\min_{\mathbf{A}} \sum_i |\hat{\mathbf{p}}_i \mathbf{A} - \hat{\mathbf{q}}_i|^2$$

Least squares (affine):

$$\mathbf{A}^* = \arg\min_{\mathbf{A}} \sum_i |\hat{\mathbf{p}}_i \mathbf{A} - \hat{\mathbf{q}}_i|^2$$

$$\frac{\partial}{\partial \mathbf{A}} \sum_i |\hat{\mathbf{p}}_i \mathbf{A} - \hat{\mathbf{q}}_i|^2 = 0 \quad \Rightarrow \quad \mathbf{A}^* = \left(\sum_i \hat{\mathbf{p}}_i^T \hat{\mathbf{p}}_i \right)^{-1} \sum_j \hat{\mathbf{p}}_j^T \hat{\mathbf{q}}_j$$

Free-form deformation

Least squares (affine):

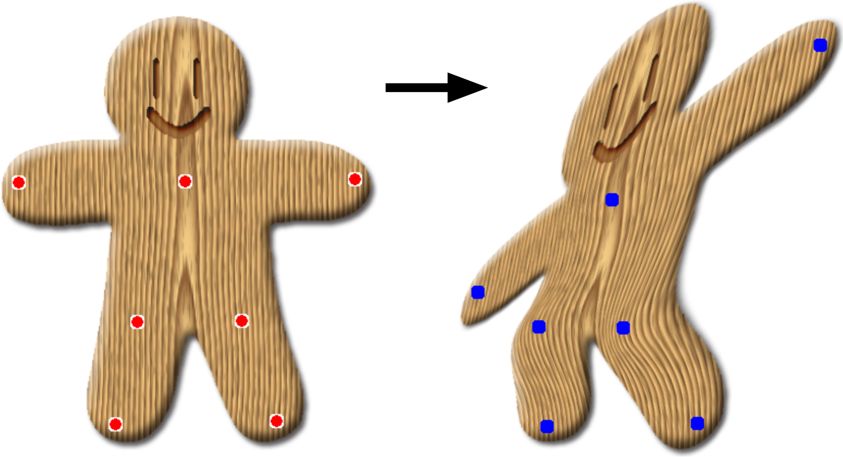
$$\mathbf{A}^* = \arg\min_{\mathbf{A}} \sum_i |\hat{\mathbf{p}}_i \mathbf{A} - \hat{\mathbf{q}}_i|^2$$

$$\frac{\partial}{\partial \mathbf{A}} \sum_i |\hat{\mathbf{p}}_i \mathbf{A} - \hat{\mathbf{q}}_i|^2 = 0 \quad \Rightarrow \quad \mathbf{A}^* = \left(\sum_i \hat{\mathbf{p}}_i^T \hat{\mathbf{p}}_i \right)^{-1} \sum_j \hat{\mathbf{p}}_j^T \hat{\mathbf{q}}_j$$

$$\mathbf{t}^* = \mathbf{q}_{\mathbf{c}} - \mathbf{p}_{\mathbf{c}} \mathbf{A}^*$$

Free-form deformation

Moving least squares (affine):



Free-form deformation

Moving least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i} \mathbf{A} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

Free-form deformation

Moving least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left|\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}\right|^{\mathbf{2}}$$

$$w_i = \frac{1}{(\mathbf{p}_i - \mathbf{v})^{2\alpha}}$$

Free-form deformation

Moving least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left|\mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i}\right|^{\mathbf{2}}$$

$$w_i = \frac{1}{(\mathbf{p}_i - \mathbf{v})^{2\alpha}}$$

$$\mathbf{A}^* = \left(\sum_i \hat{\mathbf{p}}_i^T w_i \hat{\mathbf{p}}_i\right)^{-1} \sum_j \hat{\mathbf{p}}_j^T \hat{\mathbf{q}}_j \qquad \mathbf{t}^* = \mathbf{q_c} - \mathbf{p_c}\mathbf{A}^*$$

Free-form deformation

Moving least squares (affine):

$$(\mathbf{A}^*, \mathbf{t}^*) = \arg\min_{\mathbf{A},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i}\mathbf{A} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

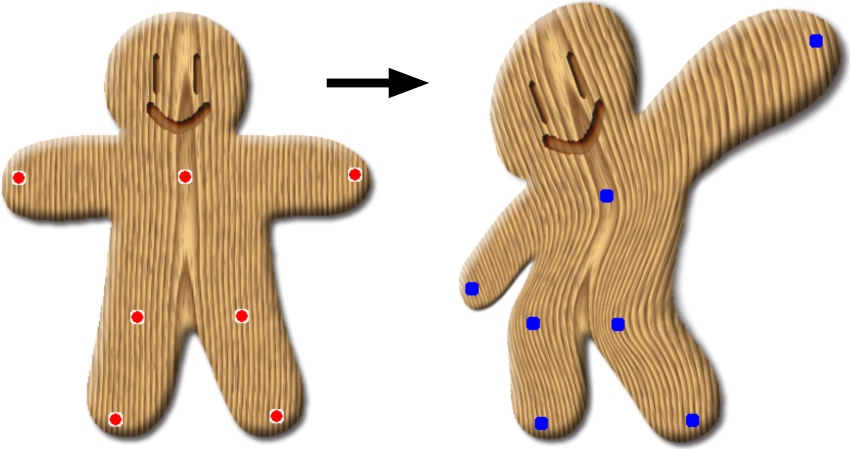
$$w_i = \frac{1}{(\mathbf{p}_i - \mathbf{v})^{2\alpha}}$$

$$\mathbf{A}^* = \left(\sum_i \hat{\mathbf{p}}_i^T w_i \hat{\mathbf{p}}_i \right)^{-1} \sum_j \hat{\mathbf{p}}_j^T \hat{\mathbf{q}}_j \qquad \mathbf{t}^* = \mathbf{q_c} - \mathbf{p_c}\mathbf{A}^*$$

$$\mathbf{p}_c = \sum_i w_i \mathbf{p}_i / \sum_i w_i \quad \mathbf{q}_c = \sum_i w_i \mathbf{q}_i / \sum_i w_i$$

Free-form deformation

Moving least squares (as-similar-as-possible):



Free-form deformation

Moving least squares (as-similar-as-possible):

$$(\mathbf{S}^*, \mathbf{t}^*) = \arg\min_{\mathbf{S},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i}\mathbf{S} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

Free-form deformation

Moving least squares (as-similar-as-possible):

$$(\mathbf{S}^*, \mathbf{t}^*) = \arg\min_{\mathbf{S},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i}\mathbf{S} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

$$\mathbf{S}^T\mathbf{S} = \lambda^2 \mathrm{I}$$

Free-form deformation

Moving least squares (as-similar-as-possible):

$$(\mathbf{S}^*, \mathbf{t}^*) = \arg\min_{\mathbf{S},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left|\mathbf{p_i}\mathbf{S} + \mathbf{t} - \mathbf{q_i}\right|^{\mathbf{2}}$$

$$\mathbf{S}^T\mathbf{S} = \lambda^2 \mathrm{I}$$

$$\mathbf{S}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^{\perp} \end{pmatrix} \left(\hat{\mathbf{q}}_i^T \quad \hat{\mathbf{q}}_i^{\perp T} \right)$$

Free-form deformation

Moving least squares (as-similar-as-possible):

$$(\mathbf{S}^*, \mathbf{t}^*) = \arg\min_{\mathbf{S},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left|\mathbf{p_i}\mathbf{S} + \mathbf{t} - \mathbf{q_i}\right|^{\mathbf{2}}$$

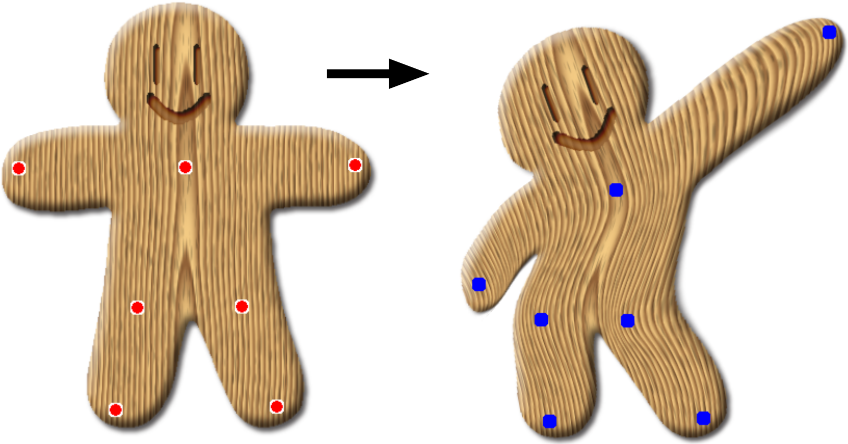
$$\mathbf{S}^T\mathbf{S} = \lambda^2 \mathrm{I}$$

$$\mathbf{S}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^\perp \end{pmatrix} \begin{pmatrix} \hat{\mathbf{q}}_i^T & \hat{\mathbf{q}}_i^{\perp T} \end{pmatrix}$$

$$\mu = \sum_i w_i \hat{\mathbf{p}}_i \hat{\mathbf{p}}_i^T \qquad (x, y)^\perp = (y, -x)$$

Free-form deformation

Moving least squares (as-rigid-as-possible):



Free-form deformation

Moving least squares (as-rigid-as-possible):

$$(\mathbf{R}^*, \mathbf{t}^*) = \arg\min_{\mathbf{R},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i} \mathbf{R} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

Free-form deformation

Moving least squares (as-rigid-as-possible):

$$(\mathbf{R}^*, \mathbf{t}^*) = \arg\min_{\mathbf{R},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i}\mathbf{R} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

$$\mathbf{R}^T\mathbf{R} = \mathrm{I}$$

Free-form deformation

Moving least squares (as-rigid-as-possible):

$$(\mathbf{R}^*, \mathbf{t}^*) = \arg\min_{\mathbf{R},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left| \mathbf{p_i}\mathbf{R} + \mathbf{t} - \mathbf{q_i} \right|^{\mathbf{2}}$$

$$\mathbf{R}^T\mathbf{R} = \mathrm{I}$$

$$\mathbf{R}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^\perp \end{pmatrix} \left(\hat{\mathbf{q}}_i^T \quad \hat{\mathbf{q}}_i^{\perp T} \right)$$

Free-form deformation

Moving least squares (as-rigid-as-possible):

$$(\mathbf{R}^*, \mathbf{t}^*) = \arg\min_{\mathbf{R},\mathbf{t}} \sum_{\mathbf{i}} \mathbf{w_i} \left|\mathbf{p_i}\mathbf{R} + \mathbf{t} - \mathbf{q_i}\right|^{\mathbf{2}}$$

$$\mathbf{R}^T\mathbf{R} = \mathrm{I}$$

$$\mathbf{R}^* = \frac{1}{\mu} \sum_i w_i \begin{pmatrix} \hat{\mathbf{p}}_i \\ \hat{\mathbf{p}}_i^{\perp} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{q}}_i^T & \hat{\mathbf{q}}_i^{\perp T} \end{pmatrix}$$

$$\mu = \sqrt{\left(\sum_i w_i \hat{\mathbf{q}}_i \hat{\mathbf{p}}_i^T\right)^2 + \left(\sum_i w_i \hat{\mathbf{q}}_i \hat{\mathbf{p}}_i^{\perp T}\right)^2}$$

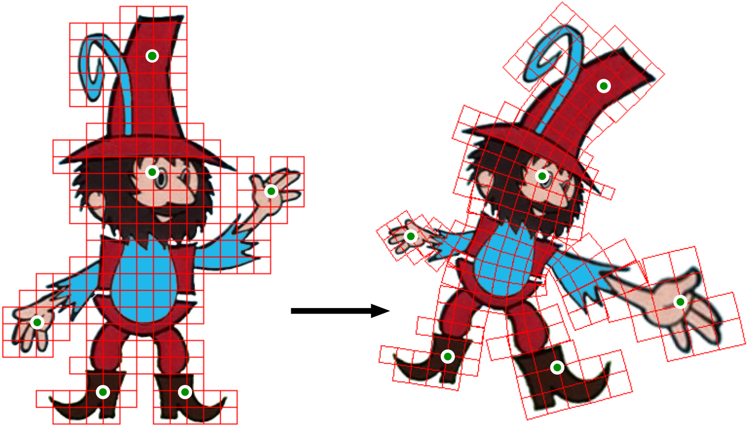
Moving least squares (advanced weights):

Moving least squares:

interactive manipulation

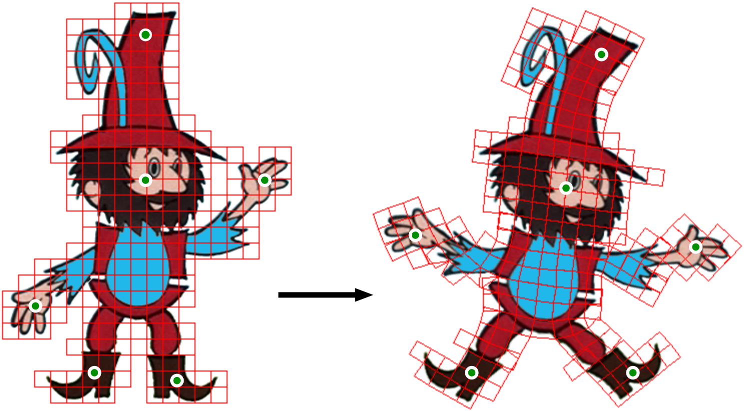
Free-form deformation

Coupled bodies (as-similar-as-possible):



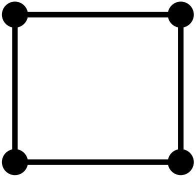
Free-form deformation

Coupled bodies (as-rigid-as-possible):

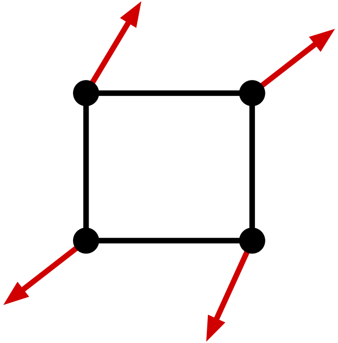


Free-form deformation

Coupled bodies (basic concept):

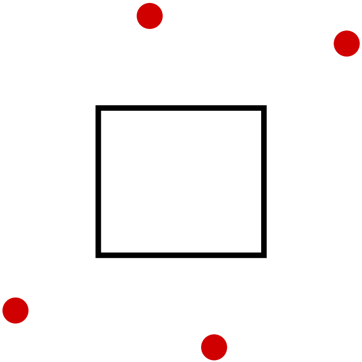


Coupled bodies (basic concept):



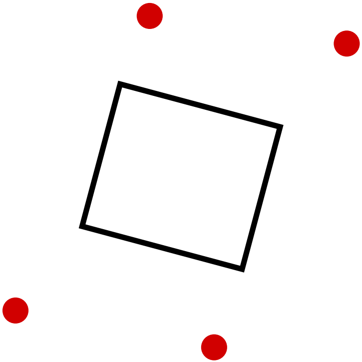
Free-form deformation

Coupled bodies (basic concept):

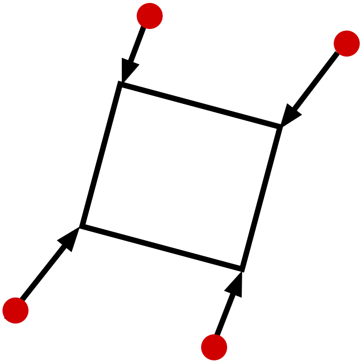


Free-form deformation

Coupled bodies (basic concept):

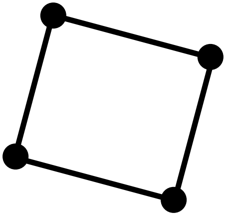


Coupled bodies (basic concept):



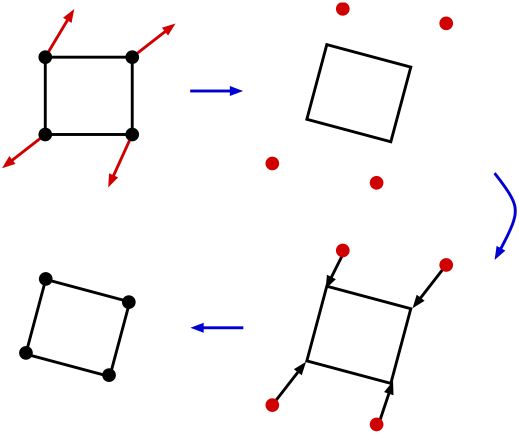
Free-form deformation

Coupled bodies (basic concept):



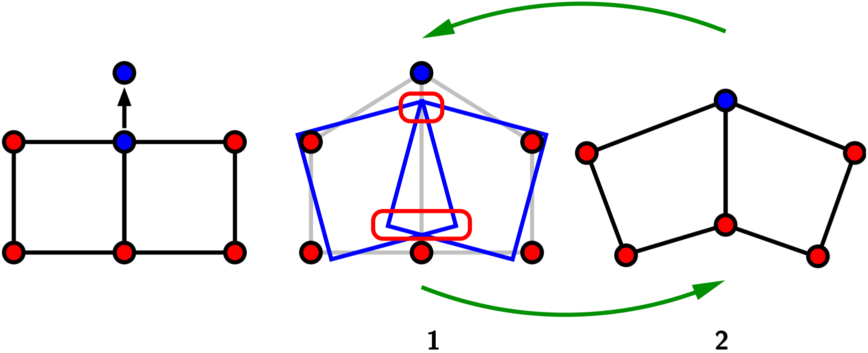
Free-form deformation

Coupled bodies (basic concept):



Free-form deformation

As-similar-as-possible coupled bodies (algorithm):

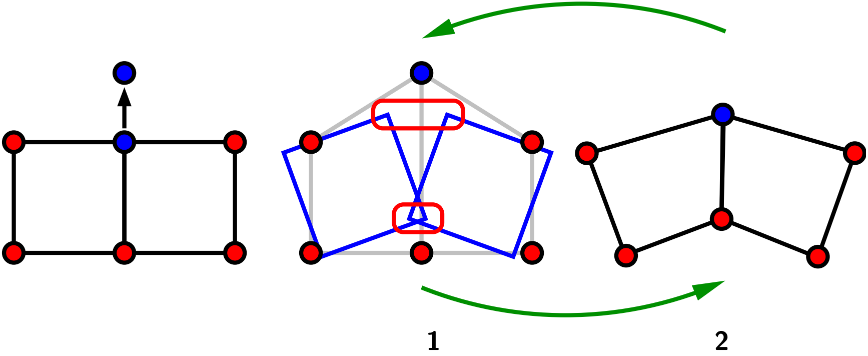


Repeat until convergence:

1. **Get best similarity transformation $(\mathrm{S}^*, \mathrm{t}^*)$ for each square.**
2. **Move each point to the centroid of its new locations.**

Free-form deformation

As-rigid-as-possible coupled bodies (algorithm):



Repeat until convergence:

1. **Get best rigid transformation $(\mathrm{R}^*, \mathrm{t}^*)$ for each square.**
2. **Move each point to the centroid of its new locations.**

Free-form deformation

As-similar-as-possible coupled bodies:

interactive manipulation

As-rigid-as-possible coupled bodies:

interactive manipulation