Bi-directional Path Tracing
and
Photon Mapping

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Main concept (global)

Real world problem
↓
Math model (model errors)
↓
Discretisation (discretisation errors, measuring errors, rouding errors)
↓
Computation (truncation errors, rounding errors)
↓
Result, result interpretation (errors estimation)

Main concept (physicaly based rendering)

Motivation, main goal: to make real images of virtual world, to outwit human senses
↓
(compromise: not all physical laws are taken into account)
↓
Math. model: the rendering equation, the volume rendering equation
↓
(compromise: the surface interpolation, discretisation, BRDF simplification, inaccurate data)
↓
Numerical method: Monte Carlo (path tracing + modifications), finite elements, …
↓
(compromise: inaccurate method, inaccurate result)
↓
Checking result: is the result still usable?

The most important topics and problems

soft shadows

color bleeding

glossy and mirror reflections

caustics

participating media

subsurface scattering

narrow passages

surface singularities

History

realistic image systhesis, first attemps: 1980 (ray tracing), 1984 (radiosity, finite elements).

ray tracing: basic [T. Whitted, 1980], Monte Carlo [Cook & al., 1984, 1986], math model [J. T. Kajiya, 1986]

various enhancements in ray tracing: [G. J. Ward, 1988], [P. Shirley, 1990], [E. P. Lafortune, 1995], [E. Veach, 1995]

photon mapping: [H. W. Jensen, 1995]

bi-directional path tracing: [E. P. Lafortune, 1993], [E. Veach, 1994]

metropolis light transport: [E. Veach, 1997]

Current development

Radiance and irradiance caching [J. Křivánek, 2006]

Matrix row-column sampling, many-light problem [M. Hašan, 2008]

Precomputed radiance transfer [Spherical Harmonic: P. P. Sloan, 2006]

…

Rendering equation [J. T. Kajiya, 1986]

Local formulation:

$$L_{\mathrm{out}}(x,\omega) = L_{\mathrm{e}}(x,\omega) + \int_{\Omega'} f_r(\omega',x,\omega) L_{\mathrm{in}}(x,\omega') \cos\theta' \, \mathrm{d}\omega'$$

where $L_{\mathrm{out}}(x,\omega)$ is an outgoing radiance from point x in direction ω,

$L_{\mathrm{in}}(x,\omega')$ is incomming radiance to point x from direction ω'

L_{e} is an emitted radiance (light sources)

f_r is the bidirectional reflection distribution function (BRDF)

θ' is the angle between normal at point x and ω'

Ω' is the unit hemisphere, $\cos\theta' > 0$

Rendering equation

Global formulation:

the substitution $\omega' \to y = \mathrm{ray}(x,\omega'),\ \mathrm{d}\omega' \to \dfrac{\cos\theta''}{\|x-y\|^2}\,\mathrm{d}s$ gives from the previous rendering equation the following one:

$$L_{\mathrm{out}}(x,\omega) = L_{\mathrm{e}}(x,\omega) + \int_{y\in S} f_r(\omega',x,\omega) L_{\mathrm{out}}(y,-\omega') V(x,y) \frac{\cos\theta' \cos\theta''}{\|x-y\|^2}\,\mathrm{d}s,$$

$$\text{where } \omega' = \frac{y-x}{\|x-y\|}, \quad V(x,y) = \begin{cases} 1 & \text{if } x\cdots y \text{ are visible} \\ 0 & \text{else} \end{cases}$$

Usable for finite elements methods, for example.

Integral as a linear operator

If we set $TL = \displaystyle\int_S f_r(\omega',\cdot,\cdot) L(y,-\omega') V(\cdot,y) \frac{\cos\theta'\cos\theta''}{\|\cdot - y\|^2}\,\mathrm{d}s,$

(where $L_{\mathrm{out}} = L$) then we get simpler form of global rendering equation:

$L = L_{\mathrm{e}} + TL$, i.e. $(I-T)L = L_{\mathrm{e}}$

The inversion of $(I-T)$ can be written by inifinite sum:

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k, \quad \text{so} \quad L = \sum_{k=0}^{\infty} T^k L_{\mathrm{e}}.$$

It means that the radiance is the sum of direct illumination + light from one reflection + light from two reflections + …

Volume rendering equation [J. T. Kajiya, 1986]

$$(\omega\cdot\nabla)L(x,\omega) = \sigma_a(x)L_{\mathrm{e}}(x,\omega) - \sigma_t(x)L(x,\omega) + \sigma_s(x)\int_{\Omega} p(\omega',x,\omega)L(x,\omega')\,\mathrm{d}\omega'$$

where ∇ is the gradient, it means that

$$(\omega\cdot\nabla)L = \omega_1\frac{\partial L}{\partial x_1} + \omega_2\frac{\partial L}{\partial x_2} + \omega_3\frac{\partial L}{\partial x_3}$$

$\sigma_{a/s}$ is absorption/scattering coefficient, $\sigma_t = \sigma_a + \sigma_s$,

p is the phase function, $\displaystyle\int_{\Omega} p(\omega,x,\omega')\,\mathrm{d}\omega' = 1$

Ω is the unit sphere.

This is the integral+differential equation.

Volume equation – a solution of differential part

Let $x = x(t) = y + \omega t$, $L(t) = L(x(t),\omega)$ then the volume rendering equation has the following form:

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} + \sigma_t(x(t))L(t) = \sigma_a(x(t))L_{\mathrm{e}}(x(t),\omega) + \sigma_s(x(t))\int_{\Omega} p(\omega',x(t),\omega)L(x(t),\omega')\,\mathrm{d}\omega'$$

The $L(t)$ makes only infinitesimal (measure 0) contribution to the integral, thus we can replace the whole right-hand side to the function of t:

$$L'(t) + u(t)L(t) = f(t)$$

this is differential quation can be solved by classical methods, but the solution has a bit impractical integral (of integral) form.

Monte Carlo Ray Tracing

advantages:

Any type of BRDF can be handled including specular reflection

Low memory consumption

Geometry can be procedural and can be duplicated

No tesselation is necessary

but disadvantages:

View-dependent calculation

Time consuming

Noise

Four times more samples needed to reach half variance

Bidirectional Path Tracing

advantages of both methods: rays from light / from the eye

two rays: x from the eye, y from light source

each vertex x_i is shaded with each vertex y_i:

$$L_{i,j}(x_i \to x_{i-1}) = f_r(y_j \to x_i \to x_{i-1})\, V(x_i, y_j)\, \frac{(y_j \to x_i) \cdot n_{x_i}}{\|x_i - y_j\|^2}\, I(y_j \to x_i)$$

$I(y_j \to x_i)$ is the radiant intensity

Radiance estimate for the pixel: $L = \sum_i \sum_j w_{i,j} L_{i,j}$.

The $w_{i,j}$ are weights with the property: $\sum_i w_{i,j} = 1$ for all j.

Bidirectional path tracing – weights

Good choice of $w_{i,j}$ is substantial

[Veach] power heuristic: $w_{i,j} = \frac{p_{i,j}^2}{\sum_{k=0}^{i+j} p_{k,i+j-k}^{\beta}}$ (most common: $\beta = 2$)

$p_{i,j}$ is probability density of existence the path $x_0, \ldots, x_i, y_j, \ldots, y_0$.

it is the product of probabilities of generating $x_i \to x_{i+1}$ (or $y_j \to y_{j+1}$).

Photon mapping – basic idea

Two pass algorithm

First pass, photon tracing: storing "photons" (vertexes of light paths),

Data of the photon: power, position, incoming direction

Photon map: storage is independent of geometry of the scene (volume oriented, kd-tree)

Second pass: rendering, paths from eye, radiance calculated using n photons around scatering point (shortest distance).

Photon mapping – radiance calculation

$$L_{\mathrm{out}}(x, \omega) = \int_{\Omega'} f_r(\omega', x, \omega) L_{\mathrm{in}}(x, \omega') \cos\theta' \, \mathrm{d}\omega'$$

$$L_{\mathrm{in}}(x, \omega') = \frac{\mathrm{d}^2 \Phi_{\mathrm{in}}(x, \omega')}{\cos\theta' \, \mathrm{d}\omega' \, \mathrm{d}s}$$

$$L_{\mathrm{out}}(x, \omega) = \int_{\Omega'} f_r(\omega', x, \omega) \frac{\mathrm{d}^2 \Phi_{\mathrm{in}}(x, \omega')}{\mathrm{d}s}$$

$$L_{\mathrm{out}}(x, \omega) \doteq \sum_{p=1}^{n} f_r(\omega_p, x, \omega) \frac{\Phi_p}{\Delta s} = \sum_{p=1}^{n} f_r(\omega_p, x, \omega) \frac{\Phi_p}{\pi r^2}$$

Photon mapping – integral evaluation (1)

$L_{\mathrm{out}} = L_{\mathrm{e}} + L_{\mathrm{ref}}$, $\quad f_r = f_S + f_D$, $\quad L_{\mathrm{in}} = L_{\mathrm{inL}} + L_{\mathrm{inC}} + L_{\mathrm{inD}}$, where:

L_{ref} is reflected radiance,

f_S is specular term and f_D is diffuse term of BRDF,

L_{inL} is direct illumination, L_{inC} is caustics and L_{inD} is other illumination

$$\begin{aligned} L_{\mathrm{ref}} = \int_{\Omega'} f_r \, L_{\mathrm{in}} \cos\theta' \, \mathrm{d}\omega' = \int_{\Omega'} f_r \, L_{\mathrm{inL}} \cos\theta' \, \mathrm{d}\omega' &+ \\ \int_{\Omega'} f_S (L_{\mathrm{inC}} + L_{\mathrm{inD}}) \cos\theta' \, \mathrm{d}\omega' &+ \\ \int_{\Omega'} f_D \, L_{\mathrm{inC}} \cos\theta' \, \mathrm{d}\omega' + \int_{\Omega'} f_D \, L_{\mathrm{inD}} \cos\theta' \, \mathrm{d}\omega' & \end{aligned}$$

Each of the integrals can be computed by different method.

Photon mapping – integral evaluation (2)

$\int_{\Omega'} f_r \, L_{\mathrm{inL}} \cos\theta' \, \mathrm{d}\omega'$ evaluated by basic ray tracing

$\int_{\Omega'} f_S (L_{\mathrm{inC}} + L_{\mathrm{inD}}) \cos\theta' \, \mathrm{d}\omega'$ evaluated by standard MC ray tracing

$\int_{\Omega'} f_D \, L_{\mathrm{inC}} \cos\theta' \, \mathrm{d}\omega'$ evaluated using special caustics photon map

$\int_{\Omega'} f_D \, L_{\mathrm{inD}} \cos\theta' \, \mathrm{d}\omega'$ evaluated using global photon map

Photon tracing – participating media

Photon tracing in participating medium: around the ray from light.

Average length d of ray before next iteration: $d = 1/\sigma_t$

Photons are stored at each point of iteration in "volume photon map".

If Rand(0,1) $\leq \sigma_s/\sigma_t$ then photon is scattered else it is absorbed.

Scattered photon: impotrance sampling by p (phase function).

Integral from volume rendering equation can be evaluated by:

$$\sigma_s(x) \int_\Omega p(\omega', x, \omega) L(x, \omega') \,\mathrm{d}\omega' \doteq \sum_{i=1}^{n} p(\omega_i, x, \omega) \frac{\Phi_i}{\frac{4}{3}\pi r^3}$$

because $L(x, \omega') = \dfrac{\mathrm{d}^2\Phi(x, \omega')}{\sigma_s(x)\,\mathrm{d}\omega'\,\mathrm{d}V}$

Photon tracing – subsurface scattering

Similar to participating media.

Rendering step: ray marching (numerical solution of differential equation)